

# A FORMULA FOR PLÜCKER COORDINATES ASSOCIATED WITH A PLANAR NETWORK

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**ABSTRACT.** For a planar directed graph  $G$ , Postnikov's boundary measurement map sends positive weight functions on the edges of  $G$  onto the appropriate totally nonnegative Grassmann cell. We establish an explicit formula for Postnikov's map by expressing each Plücker coordinate as a ratio of two combinatorially defined polynomials in the edge weights, with positive integer coefficients. In the non-planar setting, we show that a similar formula holds for special choices of Plücker coordinates.

## 1. INTRODUCTION

Totally nonnegative Grassmannians are an important subclass of general totally nonnegative homogeneous spaces, first introduced and studied by G. Lusztig and K. Rietsch (see, e.g., [6, 8, 9]). Informally speaking, the totally nonnegative Grassmannian is the part of a real Grassmann manifold where all Plücker coordinates are nonnegative. A. Postnikov's groundbreaking paper [7] established combinatorial foundations for the study of totally nonnegative Grassmannians, in particular providing the tools required for the construction of cluster algebra structures in (ordinary) Grassmannians by J. Scott [10], and for the exploration of tropical analogues by D. Speyer and L. Williams [11].

The goal of this paper is to give an explicit combinatorial formula describing the main construction in [7]: the *boundary measurement map* that assigns a point in the totally nonnegative Grassmannian to each planar directed network with positive edge weights. To state our main results, we will need to quickly recall the main features of Postnikov's construction; for complete details, see Section 2.

The construction begins with a planar directed graph  $G$  properly embedded in a disk. Every vertex of  $G$  lying on the boundary of the disk is assumed to be a source or a sink. Each edge of  $G$  is assigned a *weight*, which we treat as a formal variable. Postnikov defines the *boundary measurement matrix*  $A$  with columns labeled by the boundary vertices and rows labeled by the set  $I$  of boundary sources, as follows. Each matrix entry of  $A$  is, up to a sign that accounts for how the sources and sinks interlace along the boundary, a weight generating function for directed walks from a given boundary source to a given boundary vertex, where each walk is counted with

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a sign reflecting the parity of its topological winding index. The maximal minors  $\Delta_J(A)$  of the boundary measurement matrix  $A$  (here  $J$  is a subset of boundary vertices with  $|J| = |I|$ ) are then interpreted as Plücker coordinates of a point in a Grassmannian. The fact that these minors are nonnegative (so that we get a point in a totally nonnegative Grassmannian) follows from the assertion in [7] that each maximal minor  $\Delta_J(A)$  can be written as a *subtraction-free* rational expression in the edge weights.

Postnikov’s proof of this fact is recursive. In this paper, we provide a direct proof via an explicit combinatorial formula for the minors  $\Delta_J(A)$ , writing each of them as a ratio of two polynomials in the edge weights, with positive integer coefficients.

Here we state the main result in the case that  $G$  is *perfectly oriented*, i.e., every interior vertex of  $G$  has exactly one incoming edge or exactly one outgoing edge (or both). We will later address an extension to graphs which are planar but not necessarily perfectly oriented and discuss an analogue for non-planar perfectly oriented graphs.

In order to state our formula, we will need the following notions. A *conservative flow* in a perfectly oriented graph  $G$  is a (possibly empty) collection of pairwise vertex-disjoint oriented cycles. (Each cycle is self-avoiding, i.e., it is not allowed to pass through a vertex more than once. For perfectly oriented graphs  $G$ , this is equivalent to not repeating an edge.) For  $|J| = |I|$ , a *flow from  $I$  to  $J$*  is a collection of self-avoiding walks and cycles, all pairwise vertex-disjoint, such that each walk connects a source in  $I$  to a boundary vertex in  $J$ . (A vertex may be connected to itself by a walk with no edges.) The *weight* of a flow (conservative or not) is the product of the weights of all its edges. A flow with no edges has weight 1.

**Theorem 1.1.** *The Plücker coordinate  $\Delta_J(A)$  is given by  $\Delta_J = \frac{f}{g}$ , where  $f$  and  $g$  are nonnegative polynomials in the edge weights, defined as follows:*

- $f$  is the weight generating function for all flows from  $I$  to  $J$ ;
- $g$  is the weight generating function for all conservative flows in  $G$ .

If the underlying graph  $G$  is acyclic, then  $g = 1$ , and Theorem 1.1 reduces to the well known result of B. Lindström [5] expressing the determinant of a matrix associated with a planar acyclic network in terms of non-intersecting paths; see, e.g., [2] and references therein. Thus, Theorem 1.1 can be viewed as a generalization of Lindström’s result to non-acyclic planar networks. Another such generalization was given by S. Fomin [1] whose setup differed from Postnikov’s in that the analogues of boundary measurements did not involve any signs. In Fomin’s approach, total nonnegativity is achieved—for edge weights specialized to nonnegative real values—by writing the minors in question as formal power series with nonnegative coefficients. In contrast, Postnikov’s map produces subtraction-free expressions which are rational (that is, involve division) but finite (that is, do not require infinite summation).

The rest of the paper is organized as follows. In Section 2, we review Postnikov’s construction of boundary measurements in planar circular networks. In Section 3, we present our main result in the perfectly oriented case (Theorem 3.2, a more formal restatement of Theorem 1.1), and provide a proof based on a sign-reversing involution.

We then extend these results in two directions. In Section 4, we give a generalization of Theorem 3.2, extending to planar networks which are not necessarily perfectly oriented. In Section 5 we examine non-planar networks with perfect orientations, using G. Lawler's notion of loop-erasure in place of the topological winding index. In this generality, we show that the formula in Theorem 3.2 holds for those Plücker coordinates which are equal to individual entries in the boundary measurement matrix, but not for arbitrary minors  $\Delta_J$ .

## 2. BOUNDARY MEASUREMENTS IN PERFECTLY ORIENTED NETWORKS

**Definition 2.1.** A *planar circular directed graph* is a finite directed graph  $G$  properly embedded in a closed oriented disk (so that its edges intersect only at the appropriate vertices), together with a distinguished labeled subset  $\{b_1, \dots, b_n\}$  of *boundary vertices* such that

- (1)  $b_1, \dots, b_n$  appear in clockwise order around the boundary of the disk,
- (2) all other vertices of  $G$  lie in the interior of the disk, and
- (3) each boundary vertex  $b_i$  is incident to at most one edge.

A non-boundary vertex in  $G$  is called an *interior vertex*. Loops and multiple edges are permitted. Each boundary vertex is designated a source or a sink, even if it is an isolated vertex. We denote by  $I \subseteq [n] = \{1, \dots, n\}$  the indexing set for the boundary sources of  $G$ , so that these sources form the set  $\{b_i : i \in I\}$ .

A *planar circular network*  $N = (G, x)$  is a planar circular directed graph  $G$  together with a collection  $x = (x_e)$  of formal variables  $x_e$  labeled by the edges  $e$  in  $G$ . We call  $x_e$  the *weight* of  $e$ .

**Definition 2.2** ([7]). A planar circular directed graph (or network) is said to be *perfectly oriented* if every interior vertex either has exactly one outgoing edge (with all other edges incoming) or exactly one incoming edge (with all other edges outgoing).

For example, let  $G$  be a circular directed graph in which all interior vertices are trivalent, with no interior sources or sinks. Then  $G$  is perfectly oriented. Such a graph is shown in Figure 1; this will serve as our running example throughout Sections 2 and 3.

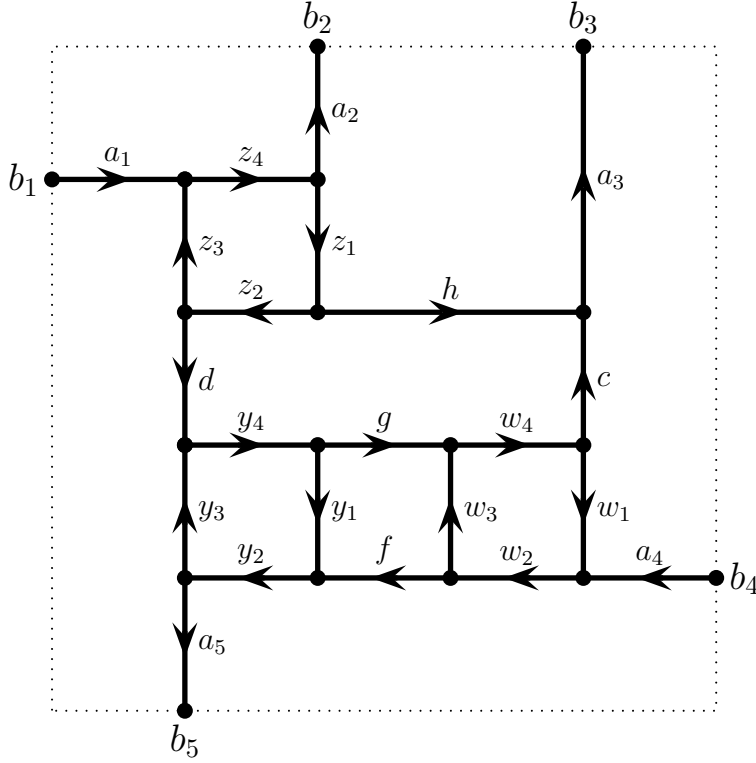
A *walk*  $P = (e_1, \dots, e_m)$  in  $G$  is formed by traversing the edges  $e_1, e_2, \dots, e_m$  in the specified order. (The head of  $e_i$  is the tail of  $e_{i+1}$ .) We write  $P : u \rightsquigarrow v$  to indicate that  $P$  is a walk starting at a vertex  $u$  and ending at a vertex  $v$ . Note that in a perfectly oriented circular graph, any self-intersecting walk between boundary vertices must repeat at least one edge at every point of self-intersection.

Define the *weight* of a walk  $P = (e_1, \dots, e_m)$  to be

$$\text{wt}(P) = x_{e_1} \cdots x_{e_m}.$$

A walk  $P : u \rightsquigarrow u$  with no edges is called a *trivial walk* and has weight 1.

**Definition 2.3** ([7]). Let  $P : u \rightsquigarrow v$  be a non-trivial walk in a planar circular directed graph  $G$  connecting boundary vertices  $u$  and  $v$ . Performing an isotopy if necessary, we may assume that the tangent vector to  $P$  at  $u$  has the same direction as the tangent vector to  $P$  at  $v$ . The *winding index*  $\text{wind}(P)$  is the signed number of full



The cycles of  $N$   
have weights  
 $W = w_1 w_2 w_3 w_4$ ,  
 $Y = y_1 y_2 y_3 y_4$ ,  
 $Z = z_1 z_2 z_3 z_4$ , and  
 $T = f y_2 y_3 y_4 g w_4 w_1 w_2$ .

FIGURE 1. A perfectly oriented planar circular network  $N$  with boundary vertices  $b_1, b_2, b_3, b_4, b_5$ . Edges are labeled by their weights.

360° turns the tangent vector makes as we travel along  $P$ , counting counterclockwise turns as positive. For a trivial walk  $P$ , we set  $\text{wind}(P) = 1$ .

**Definition 2.4** ([7]). For boundary vertices  $b_i$  and  $b_j$  in a planar circular network  $N$ , the *boundary measurement*  $M_{ij}$  is the formal power series

$$(2.1) \quad M_{ij} = \sum_{P: b_i \rightsquigarrow b_j} (-1)^{\text{wind}(P)} \text{wt}(P),$$

the sum over all directed walks  $P : b_i \rightsquigarrow b_j$ .

**Example 2.5.** In the circular network  $N$  shown in Figure 1, any walk  $P$  from  $b_1$  to  $b_2$  consists of the edges with weights  $a_1, z_4, a_2$ , together with some number of repetitions of the cycle of weight  $Z = z_4 z_1 z_2 z_3$ . Consequently,

$$M_{12} = a_1 z_4 a_2 - a_1 Z z_4 a_2 + a_1 Z^2 z_4 a_2 - a_1 Z^3 z_4 a_2 + \dots = \frac{a_1 z_4 a_2}{1 + Z}.$$

**Definition 2.6.** Let  $N$  be a planar circular network. Let  $I = \{i_1 < \dots < i_k\}$ , so that the boundary sources, listed clockwise, are  $b_{i_1}, b_{i_2}, \dots, b_{i_k}$ . The *boundary measurement matrix*  $A(N) = (a_{tj})$  is the  $k \times n$  matrix defined by

$$a_{tj} = (-1)^{s(i_t, j)} M_{i_t, j},$$

where  $s(i_t, j)$  denotes the number of elements of  $I$  strictly between  $i_t$  and  $j$ .

Let  $\Delta_J(A(N))$  denote the  $k \times k$  minor of  $A(N)$  whose columns are indexed by  $J$ . That is,  $\Delta_J(A(N)) = \det(a_{tj})_{t \in [1,k], j \in J}$ . When no confusion will arise, we may simply write  $\Delta_J$ . We note that each nontrivial boundary measurement  $M_{ij}$  (i.e. when  $i$  is a source and  $j$  is a sink) occurs as the minor  $\Delta_{I \setminus \{i\} \cup \{j\}}$ .

**Example 2.7.** Suppose that  $N$  is the planar circular network in Figure 1. Then the boundary source set is indexed by  $I = \{1, 4\}$ , and we have

$$A(N) = \begin{pmatrix} 1 & M_{12} & M_{13} & 0 & -M_{15} \\ 0 & M_{42} & M_{43} & 1 & M_{45} \end{pmatrix}.$$

**Theorem 2.8** ([7]). *If  $N = (G, x)$  is a planar circular network, then each maximal minor  $\Delta_J$  of the boundary measurement matrix can be written as a subtraction-free rational expression in the edge weights  $x_e$ .*

Postnikov's proof of Theorem 2.8 is inductive. In Sections 3 and 4, we will give an explicit combinatorial formula for the boundary measurements in a planar circular network, providing a constructive proof. Theorem 3.2 gives the formula in the perfectly oriented case, and Corollary 4.3 generalizes this result, giving a formula for an arbitrary planar circular network.

**Definition 2.9** ([7]). Let  $\pi : I \rightarrow J$  be a bijection such that  $\pi(i) = i$  for all  $i \in I \cap J$ . A pair of indices  $(i_1, i_2)$ , where  $\{i_1 < i_2\} \subset I \setminus J$ , is called a *crossing*, an *alignment*, or a *misalignment* of  $\pi$ , if the two directed chords  $[b_{i_1}, b_{\pi(i_1)}]$  and  $[b_{i_2}, b_{\pi(i_2)}]$  are arranged with respect to each other as shown in Figure 2. Define the *crossing number*  $\text{xing}(\pi)$  of  $\pi$  as the number of crossings of  $\pi$ .

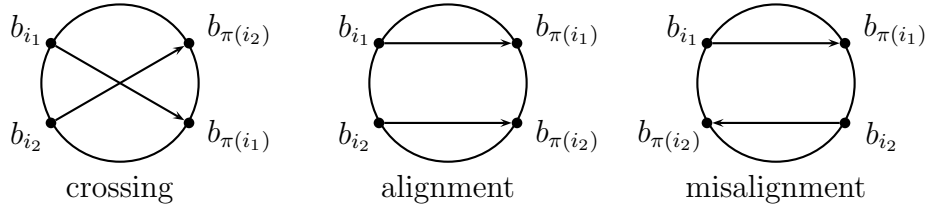


FIGURE 2. Crossings, alignments, and misalignments

**Lemma 2.10.** *For distinct  $i_1, i_2, j_1, j_2 \in [n]$ , the chords  $[b_{i_1}, b_{j_1}]$  and  $[b_{i_2}, b_{j_2}]$  cross if and only if*

$$(i_1 - j_2)(j_2 - j_1)(j_1 - i_2)(i_2 - i_1) < 0.$$

*Proof.* This is a simple verification, left to the reader. □

Lemma 2.10 immediately leads to the following corollary.

**Corollary 2.11.** *Let  $\pi : I \rightarrow J$  be a bijection such that  $\pi(i) = i$  for all  $i \in I \cap J$ . For  $\{i_1 < i_2\} \subset I \setminus J$ , the following are equivalent:*

- (1)  $(i_1, i_2)$  is a misalignment;
- (2) the chords  $[b_{i_1}, b_{i_2}]$  and  $[b_{\pi(i_1)}, b_{\pi(i_2)}]$  cross;
- (3)  $(i_1 - \pi(i_2))(\pi(i_2) - i_2)(i_2 - \pi(i_1))(\pi(i_1) - i_1) < 0$ .

We provide a new proof of the following result.

**Proposition 2.12** ([7]). *Let  $I$  index the boundary sources of a planar circular network  $N$  and let  $J \subseteq [n]$ , with  $|J| = |I|$ . Then*

$$(2.2) \quad \Delta_J(A(N)) = \sum_{\pi: I \rightarrow J} (-1)^{\text{xing}(\pi)} \prod_{i \in I} M_{i, \pi(i)},$$

the sum over all bijections  $\pi$  from  $I$  to  $J$ .

*Proof.* Taking the appropriate determinant, we see that

$$\Delta_J(A(N)) = \sum_{\pi: I \rightarrow J} (-1)^{\text{inv}(\pi)} \prod_{i \in I} (-1)^{s(i, \pi(i))} M_{i, \pi(i)},$$

where  $s(i, \pi(i))$  is defined as in Definition 2.6 and  $\text{inv}(\pi)$  is the number of inversions of  $\pi$ . Here, an *inversion* of  $\pi$  is a pair  $(i_1, i_2)$  with  $i_1 < i_2$  and  $\pi(i_1) > \pi(i_2)$ . Note that  $\prod_{i \in I} M_{i, \pi(i)} = 0$  unless  $\pi(i) = i$  for all  $i \in I \cap J$ . Thus, we wish to show that if  $\pi$  fixes the elements in  $I \cap J$ , then

$$(2.3) \quad (-1)^{\text{xing}(\pi)} = (-1)^{\text{inv}(\pi)} \prod_{i \in I} (-1)^{s(i, \pi(i))}.$$

Consider the right-hand side of (2.3). Each pair  $(i_1, i_2)$  with  $i_1 < i_2$  contributes a factor of  $\text{sgn}((\pi(i_2) - \pi(i_1)))$  to  $(-1)^{\text{inv}(\pi)}$ . Furthermore,  $i_1$  contributes a factor of  $\text{sgn}((i_1 - i_2)(i_1 - \pi(i_2))) = -\text{sgn}(i_1 - \pi(i_2))$  to  $(-1)^{s(i_1, \pi(i_1))}$ , since this product is negative if and only if  $\pi(i_2) < i_1 < i_2$ . Similarly,  $i_2$  contributes a factor of  $\text{sgn}((i_2 - i_1)(i_2 - \pi(i_1))) = -\text{sgn}((i_2 - i_1)(\pi(i_1) - i_2))$  to  $(-1)^{s(i_2, \pi(i_2))}$ . Thus, the total contribution by the pair  $(i_1, i_2)$  is

$$\text{sgn}[(i_2 - i_1)(i_1 - \pi(i_2))(\pi(i_2) - \pi(i_1))(\pi(i_1) - i_2)].$$

Taking the product over all pairs  $\{i_1 < i_2\}$ , we get  $(-1)^{\text{xing}(\pi)}$ , by Lemma 2.10.  $\square$

**Lemma 2.13.** *Let  $\pi : I \rightarrow J$  be a bijection such that  $\pi(i) = i$  for all  $i \in I \cap J$ . For  $k, l \in I \setminus J$  with  $k < l$ , let  $s_{\pi(k), \pi(l)}$  denote the transposition of the boundary vertices  $b_{\pi(k)}$  and  $b_{\pi(l)}$ , and let  $\pi^* = s_{\pi(k), \pi(l)} \circ \pi$ . Then*

$$(-1)^{\text{xing}(\pi^*)} = \begin{cases} (-1)^{\text{xing}(\pi)+1} & \text{if } (k, l) \text{ is a crossing or an alignment;} \\ (-1)^{\text{xing}(\pi)} & \text{if } (k, l) \text{ is a misalignment.} \end{cases}$$

*Proof.* Applying Lemma 2.10 and simplifying, we obtain:

$$\begin{aligned} (-1)^{\text{xing}(\pi)} (-1)^{\text{xing}(\pi^*)} &= \text{sgn} \left[ \prod_{i_1 < i_2} (i_1 - \pi(i_2))(\pi(i_2) - \pi(i_1))(\pi(i_1) - i_2) \right] \\ &\quad \cdot \text{sgn} \left[ \prod_{i_1 < i_2} (i_1 - \pi^*(i_2))(\pi^*(i_2) - \pi^*(i_1))(\pi^*(i_1) - i_2) \right] \\ &= \text{sgn} [(k - \pi(l))(\pi(l) - l)(l - \pi(k))(\pi(k) - k)], \end{aligned}$$

and the lemma follows from Corollary 2.11.  $\square$

## 3. PROOF OF THE MAIN THEOREM IN THE PERFECTLY ORIENTED CASE

Throughout this section, we assume that  $G$  is a perfectly oriented graph. Recall that  $I = \{i_1 < \dots < i_k\}$  indexes the set of boundary sources of  $G$ .

**Definition 3.1.** A subset  $F$  of (distinct) edges in a perfectly oriented planar circular directed graph  $G$  is called a *flow* if, for each interior vertex  $v$  in  $G$ , the number of edges of  $F$  that arrive at  $v$  is equal to the number of edges of  $F$  that leave from  $v$ .

A flow  $C$  is *conservative* if it contains no edges incident to the boundary. We denote by  $\mathcal{C}(G)$  the set of all conservative flows in  $G$ .

Let  $J$  be a  $k$ -element subset of  $[n]$ . We say that a flow  $F$  is a *flow from  $I$  to  $J$*  if each boundary source  $b_i$  is connected by a walk in  $F$  to a (necessarily unique) boundary vertex  $b_j$  with  $j \in J$ . If  $G$  is perfectly oriented, we denote by  $\mathcal{F}_J(G)$  the set of all flows from  $I$  to  $J$ .

The *weight* of a flow  $F$ , denoted  $\text{wt}(F)$ , is by definition the product of the weights of all edges in  $F$ . A flow with no edges has weight 1.

We note that each flow is a union of  $k$  non-intersecting self-avoiding walks, each connecting a boundary source  $b_i$  ( $i \in I$ ) to a distinct boundary vertex  $b_j$  ( $j \in J$ ), together with a (possibly empty) collection of pairwise disjoint cycles, none of which intersect any of the walks. Further, each flow lies in precisely one of the sets  $\mathcal{F}_J(G)$ . In particular, for a conservative flow, each of the  $k$  walks between boundary vertices is trivial, and  $\mathcal{C}(G) = \mathcal{F}_I(G)$ .

Using the above definitions, we can restate Theorem 1.1 as follows.

**Theorem 3.2.** *Let  $N = (G, x)$  be a perfectly oriented planar circular network. Then the maximal minors of the boundary measurement matrix  $A(N)$  are given by*

$$(3.1) \quad \Delta_J(A(N)) = \frac{\sum_{F \in \mathcal{F}_J(G)} \text{wt}(F)}{\sum_{C \in \mathcal{C}(G)} \text{wt}(C)}.$$

**Example 3.3.** Consider the planar circular network  $N$  in Figure 1, with  $I = \{1, 4\}$ . For  $J = \{1, 5\}$ , let us describe the set of flows  $\mathcal{F}_{\{1,5\}}(G)$ . The boundary vertex  $b_1$  must be connected to itself by the trivial walk  $b_1 \rightsquigarrow b_1$ . Together with the unique self-avoiding walk  $P : b_4 \rightsquigarrow b_5$  of weight  $a_4 w_2 f y_2 a_5$ , this gives a flow from  $\{1, 4\}$  to  $\{1, 5\}$ . There is one additional flow, consisting of  $P$  and the cycle of weight  $Z = z_1 z_2 z_3 z_4$  (along with the trivial walk  $b_1 \rightsquigarrow b_1$ ). Thus, the numerator of (3.1) is

$$\sum_{F \in \mathcal{F}_{\{1,5\}}(G)} \text{wt}(F) = a_4 w_2 f y_2 a_5 (1 + Z).$$

The only cycles in the network  $N$  are those of weights  $W$ ,  $Y$ ,  $Z$ , and  $T$ . Since conservative flows in  $G$  are unions of disjoint cycles, we have

$$\begin{aligned} \sum_{C \in \mathcal{C}(G)} \text{wt}(C) &= 1 + W + Y + Z + T + WZ + WY + YZ + WYZ + ZT \\ &= (1 + Z)[(1 + W)(1 + Y) + T]. \end{aligned}$$

Consequently,

$$\Delta_{\{1,5\}}(A(N)) = \frac{a_4 w_2 f y_2 a_5 (1+Z)}{(1+Z)[(1+W)(1+Y)+T]} = \frac{a_4 w_2 f y_2 a_5}{(1+W)(1+Y)+T}.$$

*Proof of Theorem 3.2.* For a bijection  $\pi : I \rightarrow J$ , let  $\mathcal{P}_\pi$  denote the set of all (possibly intersecting) collections of walks  $\mathbf{P} = (P_i)_{i \in I}$  connecting  $I$  and  $J$  in accordance with  $\pi$ :

$$\mathcal{P}_\pi = \{\mathbf{P} = (P_i : b_i \rightsquigarrow b_{\pi(i)})_{i \in I}\}.$$

In view of (2.1) and (2.2), we can rewrite the claim (3.1) as

$$(3.2) \quad \sum_{C \in \mathcal{C}(G)} \sum_{\pi: I \rightarrow J} \sum_{\mathbf{P} \in \mathcal{P}_\pi} \text{wt}(C, \mathbf{P}) = \sum_{F \in \mathcal{F}_J(G)} \text{wt}(F),$$

where  $\text{wt}(C, \mathbf{P})$ , for  $\mathbf{P} \in \mathcal{P}_\pi$ , is defined by

$$\text{wt}(C, \mathbf{P}) = \text{wt}(C) (-1)^{\text{xing}(\pi)} \prod_{i \in I} (-1)^{\text{wind}(P_i)} \text{wt}(P_i).$$

Note that if  $C$  and  $\mathbf{P}$  form a flow  $F$  from  $I$  to  $J$ , then  $\text{xing}(\pi) = 0$  and  $\text{wind}(P_i) = 0$  for all  $i$ , so that  $\text{wt}(C, \mathbf{P}) = \text{wt}(F)$ . Hence (3.2) can be restated as saying that all terms on its left-hand side cancel except for the ones for which  $C$  and  $\mathbf{P}$  form a flow from  $I$  to  $J$ . It remains to construct a sign-reversing involution proving this claim. More precisely, we need an involution  $\varphi$  on the set of pairs  $(C, \mathbf{P})$  such that

- (i)  $C \in \mathcal{C}(G)$  is a conservative flow,
- (ii)  $\mathbf{P}$  is a collection of  $k = |I|$  walks connecting  $I$  and  $J$ , and
- (iii)  $C$  and  $\mathbf{P}$  do *not* form a boundary flow.

Furthermore,  $\varphi$  must satisfy  $\text{wt}(\varphi(C, \mathbf{P})) = -\text{wt}(C, \mathbf{P})$ .

For a pair  $(C, \mathbf{P})$  satisfying (i)-(iii), we define  $\varphi(C, \mathbf{P}) = (C^*, \mathbf{P}^*)$  as follows. Let  $\mathbf{P} = (P_i)_{i \in I} \in \mathcal{P}_\pi$ , with  $\pi : I \rightarrow J$  a bijection. Choose the smallest  $i \in I$  such that  $P_i$  is not self-avoiding or has a common vertex with  $C$  or with some  $P_{i'}$  with  $i' > i$ . (Such an  $i$  exists by the assumptions we made regarding  $(C, \mathbf{P})$ .)

Let  $P_i = (e_1, \dots, e_m)$ . Choose the smallest  $q$  such that the edge  $e_q$  lies in  $C$  or in some  $P_{i'}$  with  $i' > i$ , or  $e_q = e_r$  for some  $r > q$ .

- If  $e_q$  lies in some  $P_{i'}$  with  $i' > i$ , choose the smallest such  $i'$ . (This case allows for the possibility that  $P_i$  intersects itself or  $C$  at  $e_q$ .) We will then swap the tails of  $P_i$  and  $P_{i'}$  as follows. Let  $P_{i'} = (h_1, \dots, h_{m'})$ , and choose the smallest  $q'$  such that  $h_{q'} = e_q$ . Set  $P_i^* = (e_1, \dots, e_{q-1}, e_q = h_{q'}, h_{q'+1}, \dots, h_{m'})$  and  $P_{i'}^* = (h_1, \dots, h_{q'-1}, h_{q'} = e_q, e_{q+1}, \dots, e_m)$ . Set  $\mathbf{P}^* = \mathbf{P} \setminus \{P_i, P_{i'}\} \cup \{P_i^*, P_{i'}^*\}$  and set  $C^* = C$ . (Note that  $q < \min(m, m')$  in this case, so  $\mathbf{P}^* \neq \mathbf{P}$ .)
- Otherwise we will find the first point along  $P_i$  where we can move a cycle from  $C$  to  $P_i$  or vice versa, as follows. If  $P_i$  is not self-avoiding, let  $\ell$  be the first cycle that  $P_i$  completes. That is, choose the smallest  $s$  such that  $e_r = e_s$  for some  $r < s$ ; then  $\ell = (e_r, e_{r+1}, \dots, e_{s-1})$ . If  $P_i$  is self-avoiding, then set  $s = \infty$ . If  $C$  intersects  $P_i$ , choose the smallest  $t$  such that  $e_t$  occurs in a (necessarily unique) cycle  $L = (l_1, l_2, \dots, l_w)$  in  $C$ , where  $l_1 = e_t$ . If  $C \cap P_i = \emptyset$ , then set  $t = \infty$ . Note that at least one of  $t$  or  $s$  must be finite, and  $t \neq s$ , since  $e_s = e_r$  and  $r < s$ .



- If  $t < s$ , we move  $L$  from  $C$  to  $P_i$ , as follows. Set  $C^* = C \setminus \{L\}$ ,  $P_i^* = (e_1, \dots, e_{t-1}, e_t = l_1, \dots, l_w, e_t, \dots, e_m)$ , and  $\mathbf{P}^* = \mathbf{P} \setminus \{P_i\} \cup \{P_i^*\}$ .
- If  $t > s$ , we move  $\ell$  from  $P_i$  to  $C$ , as follows. Set  $C^* = C \cup \{\ell\}$ ,  $P_i^* = (e_1, \dots, e_{r-1}, e_s, \dots, e_m)$ , and  $\mathbf{P}^* = \mathbf{P} \setminus \{P_i\} \cup \{P_i^*\}$ .

It is easy to see that, with this definition, the image  $(C^*, \mathbf{P}^*)$  is again a pair of the required kind, i.e., it satisfies the conditions (i)-(iii) above.

Let us verify that  $\varphi$  is an involution. First, we check that  $\varphi$  does not change the value of  $i$ . That is, among all walks in  $\mathbf{P}^*$  which intersect themselves, another walk, or a cycle in  $C^*$ , the walk with the smallest index (of its starting point) is  $P_i^*$ . Indeed, our moves only affect  $P_i$ ,  $P_{i'}$ , and  $C$ , keeping their combined set of edges intact, so the involution will not introduce a new self-intersection in any  $P_a$  with  $a < i$ , nor will it introduce an intersection between  $P_a$  and any path or cycle.

Consider  $\varphi(C, \mathbf{P}) = (C, \mathbf{P}^*)$  in the first case. After swapping tails,  $P_i^*$  still has no intersections with  $C$  or any of the other paths before the edge  $e_q$ . Further,  $P_i^*$  does not have any self-intersections before  $e_q$  (though it may have self-intersections at  $e_q$ ), since  $P_i$  did not have any self-intersections before  $e_q$  and the tail of  $P_{i'}$  did not intersect  $P_i$  before  $e_q$ . Thus,  $e_q$  remains the first edge along  $P_i^*$  with an intersection. Now,  $P_i^*$  and  $P_{i'}$  intersect at this edge, and no path with smaller index intersects  $P_i^*$  at  $e_q$ , so we will swap the same tails again.

Consider the second case, with  $\varphi(C, \mathbf{P}) = (C \setminus \{L\}, \mathbf{P}^*)$  or  $\varphi(C, \mathbf{P}) = (C \cup \{\ell\}, \mathbf{P}^*)$ . Here,  $P_i$  intersects itself or  $C$  at  $e_q$ , but does not intersect any other path at  $e_q$ . After moving a cycle, the same is true for  $P_i^*$ . (If the cycle moved starts at  $e_q$ , then either a self-intersection becomes an intersection with  $C$ , or an intersection with  $C$  becomes a self-intersection. If the cycle moved starts later, then the intersections at  $e_q$  remain as they are.) If  $P_i$  intersects  $C$  before completing its first cycle, then  $P_i^*$  will complete its first cycle before intersecting  $C \setminus \{L\}$ . If  $P_i$  completes its first cycle  $\ell$  before intersecting  $C$ , then  $P_i^*$  will intersect  $C \cup \{\ell\}$  before completing its first cycle. Thus, the same cycle is moved both times. We have now shown that  $\varphi$  is an involution.

Finally, we verify that  $\varphi$  is sign-reversing. In the case of tail swapping, we need to show that  $\text{wind}(P_i) + \text{wind}(P_{i'}) + \text{xing}(\pi) + \text{wind}(P_i^*) + \text{wind}(P_{i'}^*) + \text{xing}(\pi^*)$  is odd, where  $\pi^*$  is the bijection such that  $\mathbf{P}^* \in \mathcal{P}_{\pi^*}$ . By Lemma 2.13,  $\text{xing}(\pi) + \text{xing}(\pi^*)$  is even if and only if  $(i, i')$  is a misalignment. Thus, we need to show that  $(i, i')$  is a misalignment if and only if

$$(3.3) \quad \text{wind}(P_i) + \text{wind}(P_{i'}) + \text{wind}(P_i^*) + \text{wind}(P_{i'}^*) \equiv 1 \pmod{2}.$$

This statement is in fact true for *any* instance of tail swapping, i.e., it does not rely on our particular choice of the walks  $P_i$  and  $P_{i'}$  sharing an edge  $e_q$ . Viewing (3.3) as a purely topological condition, we can “unwind” each of the 4 subwalks from which our walks  $P_i$  and  $P_{i'}$  are built, keeping  $e_q$  fixed. This will not change the parity in (3.3) since each loop contained entirely in one of the initial or terminal subwalks will contribute twice, once to  $\text{wind}(P_i) + \text{wind}(P_{i'})$  and once to  $\text{wind}(P_i^*) + \text{wind}(P_{i'}^*)$ .

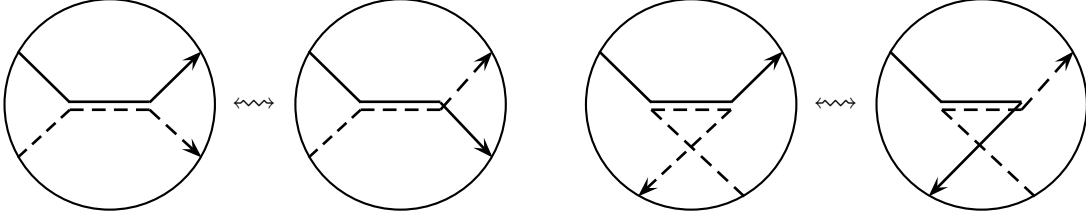


FIGURE 3. Winding index and tail swapping

Deforming the walks as necessary, we then obtain one of the four pictures shown in Figure 3. The last two of the four pictures represent misalignments, and indeed, these are precisely the two cases in which (3.3) holds.

In the remaining case (moving a cycle from  $C$  to  $\mathbf{P}$  or vice versa),  $\text{wind}(P_i)$  changes parity, while  $\text{xing}(\pi)$  and all other winding numbers do not change. Hence  $\text{wt}(\varphi((C, \mathbf{P})) = -\text{wt}(C, \mathbf{P})$ , as desired.  $\square$

#### 4. EXTENDING TO PLANAR GRAPHS WITH ARBITRARY ORIENTATIONS

In this section, we provide an extension of Theorem 3.2 for arbitrarily oriented planar networks. The proof relies on Postnikov's process in [7] for transforming an arbitrary planar circular network into a partially specialized perfectly oriented planar circular network.

When  $G$  is a perfectly oriented graph, the following definition is equivalent to Definition 3.1. This extension provides the appropriate setup for working in arbitrarily oriented graphs  $G$ .

**Definition 4.1.** A subset  $F$  of (distinct) edges in a planar circular directed graph  $G$  (not necessarily perfectly oriented) is called an *alternating flow* if, for each interior vertex  $v$  in  $G$ , the edges  $e_1, \dots, e_d$  of  $F$  which are incident to  $v$ , listed in clockwise order around  $v$ , alternate in orientation (that is, directed towards  $v$  or directed away from  $v$ ).

In an alternating flow  $F$ , define the walks  $W_i$  (with  $i \in I$ ) as follows. If  $b_i$  is isolated in  $F$ , set  $W_i$  to be the trivial walk from  $b_i$  to itself. Otherwise, let  $W_i$  be the unique path leaving  $b_i$  which, at each subsequent vertex, takes the first left turn in  $F$ , until it arrives at another boundary vertex.

For a  $k$ -element subset  $J$  of  $[n]$ , we say that an alternating flow  $F$  is a *flow from  $I$  to  $J$*  if each boundary source  $b_i$  is connected by  $W_i$  to a boundary vertex  $b_j$  with  $j \in J$ . (The vertices  $b_j$  are necessarily distinct.) Let  $\mathcal{A}_J$  denote the set of alternating flows from  $I$  to  $J$ . In particular,  $\mathcal{A}_I$  is precisely the set of conservative alternating flows.

**Definition 4.2.** Suppose  $F$  is an alternating flow. For each vertex  $v$  in  $G$ , let  $\tau(v, F)$  denote the number of edges of  $F$  coming into  $v$ . Set

$$\theta(F) = \sum_v \max\{\tau(v, F) - 1, 0\}.$$

**Corollary 4.3.** *Let  $N = (G, x)$  be a planar circular network with source set indexed by  $I$ . Then the maximal minors of the boundary measurement matrix  $A(N)$  are given by the formula*

$$(4.1) \quad \Delta_J(A(N)) = \frac{\sum_{F \in \mathcal{A}_J(G)} 2^{\theta(F)} \text{wt}(F)}{\sum_{C \in \mathcal{A}_I(G)} 2^{\theta(C)} \text{wt}(C)}.$$

The proof of Corollary 4.3 will follow Proposition 4.4, which describes Postnikov's transformation process in detail.

Let  $E(G)$  denote the edge set of  $G$ . Let  $\Delta_J(A(N))(\alpha)$  denote the evaluation of the subtraction-free rational expression  $\Delta_J(A(N))$  under a specialization map  $\alpha : E(G) \rightarrow \mathbb{R}$  assigning a positive real weight  $\alpha_e$  to each edge  $e$ .

For boundary measurements, there is no loss of generality in assuming that  $G$  has no internal sources or sinks. Further, we may assume that there are no vertices of degree 2. Indeed, if there is a vertex  $v$  with exactly one incoming edge  $e_1$  and exactly one outgoing edge  $e_2$ , we may remove  $v$  and glue  $e_1$  and  $e_2$  into a single edge  $e$  of weight  $x_e = x_{e_1}x_{e_2}$ .

**Proposition 4.4** ([7]). *Let  $N = (G, x)$  be a planar network with boundary sources indexed by  $I$  and positive weight function  $\alpha : E(G) \rightarrow \mathbb{R}$ , with  $x_e \mapsto \alpha_e$ . Let  $N' = (G', x')$  and  $\alpha' : E(G') \rightarrow \mathbb{R}$  (with  $x'_e \mapsto \alpha'_e$ ) denote a perfectly oriented planar network and corresponding positive weight function obtained from  $N$  and  $\alpha$  by the process described below. Then for all  $J \subset [n]$  with  $|J| = |I|$ , we have*

$$\Delta_J(A(N))(\alpha) = \Delta_J(A(N'))(\alpha').$$

To obtain  $N'$  from  $N$ , we perform the following operations in stages (1)-(3).

- (1) First, suppose that  $N$  has an internal vertex  $v$  of degree greater than 3; let  $e_1, \dots, e_d$  denote the edges incident to  $v$ , listed in clockwise order. If two adjacent edges  $e_i$  and  $e_{i+1}$  (modulo  $n$ ) have the same orientation, either both towards  $v$  or both away from  $v$ , we choose such a pair, pull these edges away from  $v$ , insert a new vertex  $v'$  and a new edge  $e$  (directed from  $v'$  to  $v$  when  $e_i$  and  $e_{i+1}$  are edges entering  $v$  and from  $v$  to  $v'$  when  $e_i$  and  $e_{i+1}$  are edges leaving  $v$ ), and attach the edges  $e_i$  and  $e_{i+1}$  to  $v'$ . (See Figure 4.) We set  $\alpha'_e = 1$ . Repeat until the resulting network has no vertices  $v$  of this form.

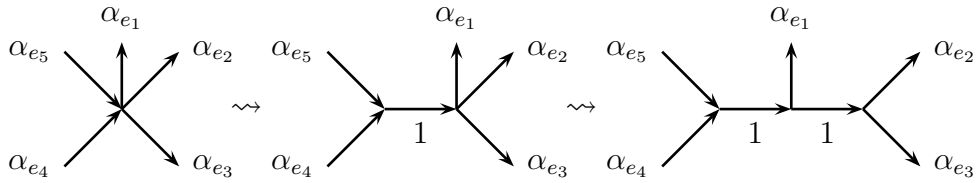


FIGURE 4. Pulling out adjacent edges with the same orientation.

- (2) If a vertex  $v$  of degree greater than 3 remains, its incident edges must alternate orientation in clockwise order. In this case we blow up the vertex  $v$  into a cycle with edges all oriented clockwise, as in Figure 5. If  $e$  is an edge coming into  $v$ , we set  $\alpha'_e = 2\alpha_e$ , and if  $e$  is one of the new edges created to make the cycle, we set  $\alpha'_e = 1$ . Repeat until the resulting network has no vertices  $v$  of this form.

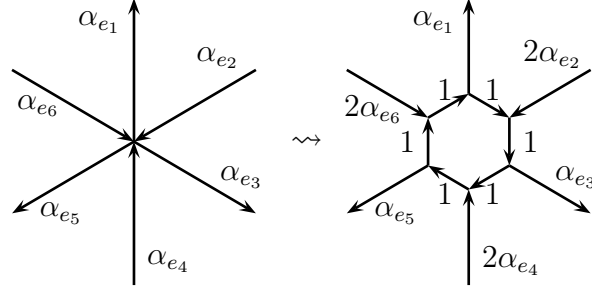


FIGURE 5. Blowing up a vertex with alternating edge directions.

- (3) Finally, for any remaining edge  $e$  unaffected by these steps (i.e. such that  $\alpha'_e$  has not yet been specified), set  $\alpha'_e = \alpha_e$ . Let  $N'$  and  $\alpha'$  denote the final result.

By contracting an edge  $e$ , we mean removing the edge  $e$  and identifying its two endpoints. (If we contract all edges in a connected subset of edges, the image is a single vertex.) It is easy to see that by contracting all new edges created in Proposition 4.4 above, we obtain  $G$  from  $G'$ .

**Definition 4.5.** Let  $B(G)$  denote the set of vertices of  $G$  around which the orientations of edges switch at least four times. Call such a vertex  $v$  a *blowup vertex*. These are precisely the vertices which are blown up into cycles in the second stage of Proposition 4.4.

For an alternating flow  $F$  in a planar network  $N$ , we define  $\epsilon(F)$  to be the number of edges of  $F$  which enter a blowup vertex of  $G$ ,  $\beta(F)$  to be the number of blowup vertices of  $G$  which occur as the endpoint of some edge in  $F$ , and  $\eta(F)$  to be the number of blowup vertices of  $G$  which are not endpoints of any edge in  $F$ . Recalling Definition 4.2, note that  $\theta(F) = \epsilon(F) - \beta(F)$ .

*Proof of Corollary 4.3.* Fix an image  $N'$  of  $N$  under the transformation in Proposition 4.4, and let  $F'$  be an alternating flow in  $N'$ . It is easily verified that contracting all edges in  $E(G') - E(G)$  gives a bijection between alternating flows  $F'$  in  $G'$  and pairs  $(F, A)$ , where  $F$  is an alternating flow in  $G$  and  $A$  is a subset of vertices in  $B(G)$  which are not endpoints of any edges in  $F$ . Further, extending  $\alpha$  and  $\alpha'$  linearly, we have

$$\alpha'(\text{wt}(F')) = 2^{\epsilon(F)} \alpha(\text{wt}(F)),$$

and there are  $2^{\eta(F)}$  flows  $F'$  corresponding to a given flow  $F$ , all with the same weight.

Since this relationship holds for every positive specialization  $\alpha$ , Theorem 3.2 and Proposition 4.4 then imply that

$$\Delta_J(A(N)) = \frac{\sum_{F \in \mathcal{A}_J(G)} 2^{\epsilon(F) + \eta(F)} \text{wt}(F)}{\sum_{C \in \mathcal{A}_I(G)} 2^{\epsilon(C) + \eta(C)} \text{wt}(C)}.$$

Cancelling a factor of  $|B(G)| = \eta(F) + \beta(F)$  from each term in the numerator and denominator, we obtain the formula (4.1), as desired.  $\square$

## 5. NOTES ON PLÜCKER COORDINATES FOR PERFECTLY ORIENTED NON-PLANAR NETWORKS

It is natural to ask to what extent we can develop these constructions in the non-planar setting. While the notion of the topological winding index only makes sense for planar graphs, Lawler's notion of loop-erasure in [4] allows us to give a non-planar analogue of the winding index if  $G$  is perfectly oriented. In this non-planar setting, we no longer have the positivity results, but we can describe those Plücker coordinates which are individual boundary measurements.

We begin by extending the definition of circular directed graphs and networks (Definition 2.1) to suit the non-planar setting. For a general *circular directed graph*, we no longer require that  $G$  has a planar embedding in a disk, but we still ask for the boundary vertices to be labeled in cyclic order and for each boundary vertex to be adjacent to at most one edge.

**Definition 5.1** ([1, 3]). The *loop-erased part* of a walk  $P : b_i \rightsquigarrow b_j$ , denoted  $\text{LE}(P)$ , is defined recursively as follows. If  $P = (e_1, \dots, e_m)$  does not have any self-intersections, then  $\text{LE}(P) = P$ . Otherwise, we set  $\text{LE}(P) = \text{LE}(P')$ , where  $P'$  is obtained from  $P$  by removing the first cycle it completes. More precisely, when  $G$  is perfectly oriented, find the smallest value of  $s$  such that there exists  $r < s$  with  $e_r = e_s$ , and remove the segment  $e_r, e_{r+1}, \dots, e_{s-1}$  from  $P$  to obtain  $P'$ . The *loop-erasure number*  $\text{loop}(P)$  is defined as the number of cycles erased during the calculation of  $\text{LE}(P)$ . With the notation as above, we have  $\text{loop}(P) = \text{loop}(P') + 1$ , and  $\text{loop}(P) = 0$  when  $P$  is a self-avoiding walk.

**Proposition 5.2** ([7]). *Suppose that  $G$  is a perfectly oriented planar circular directed graph. If  $P$  is a walk from a boundary vertex  $b_i$  to a boundary vertex  $b_j$ , then  $(-1)^{\text{loop}(P)} = (-1)^{\text{wind}(P)}$ .*

*Proof.* Each boundary vertex is incident to at most one edge, so  $P$  has no self-intersections at its endpoints. Since  $G$  is perfectly oriented,  $P$  repeats at least one edge at every self-intersection. The claim then follows by induction on  $\text{loop}(P)$ , as an erasure of a cycle changes the winding index by  $\pm 1$ .  $\square$

Proposition 5.2 allows us to view  $\text{loop}(P)$  as a natural generalization of  $\text{wind}(P)$  for the non-planar case. This observation leads to an extension of Postnikov's construction (which applies to planar networks and employs the winding index) to arbitrary

perfectly oriented graphs. Definitions 2.2, 2.3, 2.4, 2.6, 2.9, and 3.1 then extend to perfectly oriented non-planar networks in the obvious way, replacing  $\text{wind}(P)$  with  $\text{loop}(P)$  wherever appropriate.

**Corollary 5.3.** *Suppose  $N = (G, x)$  is a perfectly oriented circular network with boundary source set indexed by  $I$ . Then, for  $i \in I$  and  $j \in [n]$ , we have*

$$M_{ij} = \Delta_{(I \setminus \{i\}) \cup \{j\}}(A(N)) = \frac{\sum_{F \in \mathcal{F}_{(I \setminus \{i\}) \cup \{j\}}(G)} \text{wt}(F)}{\sum_{C \in \mathcal{C}(G)} \text{wt}(C)}.$$

*Proof.* This follows directly from the proof of Theorem 3.2, since for these special Plücker coordinates, the tail swapping process of the proof is never called upon.  $\square$

Although the result holds for those minors  $\Delta_J$  which are boundary measurements  $M_{ij}$ , it is generally not valid for the remaining Plücker coordinates. For non-planar networks, tail swapping does not always yield the sign change in  $(-1)^{\text{xing}}$  that we obtain in the planar case. As a result, the numerator and denominator of a minor  $\Delta_J$  do not necessarily simplify to linear combinations of flow weights.

**Example 5.4.** Consider the network  $N$  in Figure 6, with boundary measurement matrix  $A(N)$  below. The minors  $\Delta_{12}$ ,  $\Delta_{13}$ ,  $\Delta_{14}$ ,  $\Delta_{24}$ , and  $\Delta_{34}$  all satisfy Theorem 3.2.

$$A(N) = \begin{pmatrix} 1 & \frac{a_1 f a_2}{1 + c d e f} & \frac{a_1 f e d a_3}{1 + c d e f} & 0 \\ 0 & \frac{a_4 d c f a_2}{1 + c d e f} & \frac{a_4 d a_3}{1 + c d e f} & 1 \end{pmatrix}$$

However, for  $\Delta_{23}$ , we do not get the desired cancellation; in the simplified rational expression, both the numerator and denominator are quadratic in flow weights. We have

$$\Delta_{23}(A(N)) = \frac{a_1 a_2 a_3 a_4 \cdot d f (1 - c d e f)}{(1 + c d e f)^2}.$$

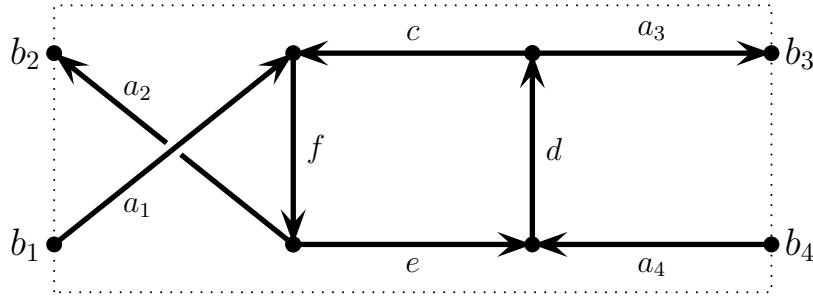


FIGURE 6. Boundary measurements in a non-planar network.

**Remark 5.5.** If we consider flow weights as polynomials with coefficients in the finite field of two elements,  $\mathbb{F}_2$ , then (3.1) holds for all  $\Delta_J$  in the perfectly oriented non-planar case; this also follows directly from the proof of Theorem 3.2.

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